Polynomial Heisenberg algebras and higher order supersymmetry

David J. Fernández C.[a], and Véronique Hussin[b]

[a]Depto Física, CINVESTAV, AP 14-740, 07000 México DF, Mexico; [b]Département de Mathématiques, Université de Montréal, CP 6128, Succ Centre-Ville, Montréal, H3C 3J7 Canada

Abstract. It is shown that the higher order supersymmetric partners of the harmonic oscillator Hamiltonian provide the simplest non-trivial realizations of the polynomial Heisenberg algebras. A linearized version of the corresponding annihilation and creation operator leads to a Fock representation which is the same as for the harmonic oscillator Hamiltonian.

1. Introduction. The non-linear algebras have attracted attention in recent years due to the fact that they start to be applied in physics. Those structures arise as deformations of a Lie algebra in which some commutation relations are replaced by non-linear functions of the generators [1-5]. Of particular interest are the simplest deformations of the standard Heisenberg-Weyl algebra (the so-called polynomial Heisenberg algebras) in which the commutator of the annihilation and creation operators becomes a polynomial in the Hamiltonian but at the same time it commutes with those ladder operators as in the standard case [6-11]. Let us notice that these modified algebraic relations provide information about the spectrum, which turns out to be a variant of the equally spaced levels of the harmonic oscillator. Concrete realizations of those algebras arise if the annihilation and creation operators become differential operators of order greater than one [6,12]. The simplest non-trivial case of such a realization is provided by considering the natural pair of annihilation and creation operators for the Hamiltonians generated by means of the higher order supersymmetric quantum mechanics (HSUSY QM) applied to the harmonic oscillator [11]. This will be the main subject discussed in this paper. In order to do that, firstly we will present some generalities of the polynomial Heisenberg algebras. Then, we will proceed by discussing the standard SUSY and HSUSY QM as a mechanism to generate solvable partner potentials from a given initial one [11,13-26]. From that procedure it will be obvious the existence and the corresponding structure of a pair of natural creation and annihilation operators for the HSUSY partners of the oscillator potential, a very original construction proposed by Mielnik for the first order SUSY [27-28]. It will be shown that those operators realize in a simple way the polynomial Heisenberg algebras. Later on it will be constructed a 'linearized' version of those algebras in which the modified ladder operators act onto the energy eigenstates as the standard generators do in the Heisenberg-Weyl linear case [11,29]. We will finish the paper with some general conclusions and an outlook for future work.

2. Polynomial Heisenberg algebras. Let us remember the standard algebra of the harmonic oscillator:

$$[H, a] = -a, \quad [H, a^\dagger] = a^\dagger, \tag{1}$$

$$[a, a^\dagger] = 1. \tag{2}$$

In the coordinates representation the operators $\{H, a, a^\dagger\}$ are given by:

$$H = -\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}x^2, \tag{3}$$

$$a=\frac{1}{\sqrt{2}}\left(\frac{d}{dx}+x\right),\quad a^\dagger=\frac{1}{\sqrt{2}}\left(-\frac{d}{dx}+x\right),\tag{4}$$

the number operator reads:

$$N=a^\dagger a,\tag{5}$$

and there is a *linear* dependence between N and H:

$$N=H-\tfrac{1}{2}\equiv N(H).\tag{6}$$

The polynomial Heisenberg algebras of $(n-1)$-th order are deformations of the above algebra in which the two commutation relationships (1) are maintained [6-11]:

$$[H,L]=-L,\quad [H,L^\dagger]=L^\dagger,\tag{7}$$

but equation (2) is substituted by:

$$[L,L^\dagger]\equiv N(H+1)-N(H).\tag{8}$$

The generalized number operator, defined by $N=L^\dagger L\equiv N(H)$, becomes now a n-th order polynomial in the Hamiltonian H:

$$N(H)=\prod_{i=1}^{n}\left(H-\mathcal{E}_i\right),\tag{9}$$

and $L,\ L^\dagger$ are realized by n-th order differential operators. Let us notice that $[L,L^\dagger]$ is a $(n-1)$-th order polynomial in H.

The algebraic properties of $\{H,L,L^\dagger\}$ provide already some information about the spectrum of H. In order to see that, let us analyze the space of solutions of the n-th order differential equation (the kernel K_L of L):

$$L\,\psi=0.\tag{10}$$

Notice that K_L is invariant under H. This and the fact that any solution to (10) obeys:

$$L^\dagger L\,\psi=\prod_{i=1}^{n}\left(H-\mathcal{E}_i\right)\psi=0,\tag{11}$$

suggest to select as the basis of K_L those solutions which are simultaneously eigenstates of H with eigenvalues $\mathcal{E}_i$:

$$H\psi_{\mathcal{E}_i}=\mathcal{E}_i\psi_{\mathcal{E}_i}.\tag{12}$$

However, some of the eigenfunctions $\psi_{\mathcal{E}_i}$ could be non-normalizable. Let us suppose that s of them, $\{\psi_{\mathcal{E}_i}, i=1,\dots,s\}$, are normalizable, which guarantee that they are orthogonal. Taking those as *extremal* states one can construct, by means of the iterated action of $L^\dagger$, s energy ladders of infinite length and spacing $\Delta E=1$, each one of them starting from $\mathcal{E}_i$ (see Figure 1).

It could happen, however, that for the ladder starting from $\mathcal{E}_j$ there exists a natural number l such that:

$$\left(L^\dagger\right)^{l-1}\psi_{\mathcal{E}_j}\neq 0,\quad \left(L^\dagger\right)^{l}\psi_{\mathcal{E}_j}=0.\tag{13}$$

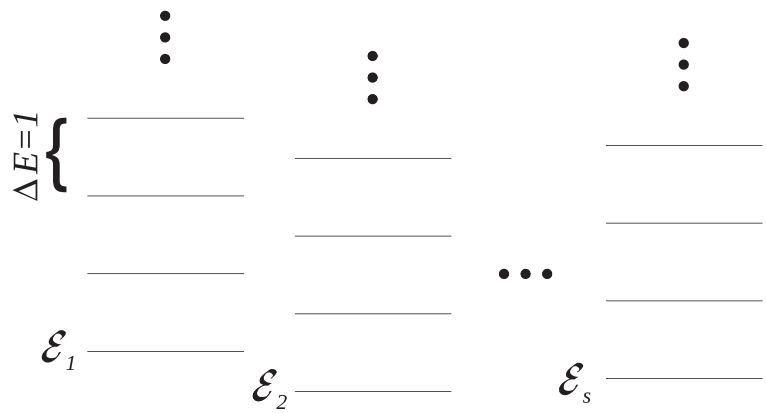

Figure 1: Possible spectra for the Hamiltonians satisfying equations (7-9). There are s extremal states from which the s infinite ladders are constructed through the iterated action of $L^\dagger$.

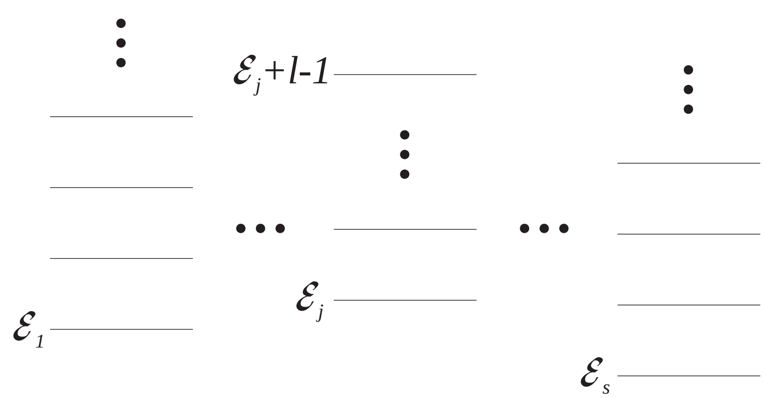

Figure 2: The same as in figure 1 excepting that the j-th ladder is finite due to the fact that $(L^\dagger)^l \psi_{\mathcal{E}_j} = 0$.

In such a case, by multiplying $(L^\dagger)^l\psi_{\mathcal{E}_j}$ to the left by L, it is seen that one of the remaining roots of (11), $\{\mathcal{E}_i, i=s+1,\dots,n\}$ is of kind $\mathcal{E}_k=\mathcal{E}_j+l$, $k\in\{s+1,\dots,n\}$, $j\in\{1,\dots,s\}$. Thus, the spectrum of H will consist of $s-1$ infinite ladders and a finite one of length l, starting from $\mathcal{E}_j$ and ending at $\mathcal{E}_j+l-1$. An scheme of this situation is represented in figure 2. This discussion means that the H of equations (7-9) is not, in general, the harmonic oscillator Hamiltonian.

3. Standard supersymmetric quantum mechanics. The standard SUSY algebra with two generators $Q_1,\ Q_2$, as introduced by Witten, reads [13]:

$$\begin{aligned} &[Q_i, H_{\rm ss}] = 0, \\ &\{Q_i, Q_j\} = \delta_{ij} H_{\rm ss}. \end{aligned} \tag{14}$$

The simplest realization of such an algebra is of kind:

$$\begin{aligned} &Q_1 = \frac{1}{\sqrt{2}}(Q^\dagger + Q), \quad Q_2 = \frac{1}{i\sqrt{2}}(Q^\dagger - Q), \\ &Q = \begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix}, \quad Q^\dagger = \begin{pmatrix} 0 & B^\dagger \\ 0 & 0 \end{pmatrix}, \quad H_{\rm ss} = \begin{pmatrix} B^\dagger B & 0 \\ 0 & BB^\dagger \end{pmatrix}, \end{aligned} \tag{15}$$

where usually $B^\dagger$ is a first order differential operator

$$B^\dagger \equiv A_1^\dagger = \frac{1}{\sqrt{2}}\left[-\frac{d}{dx} + \alpha_1(x,\epsilon)\right], \quad B = (B^\dagger)^\dagger, \tag{16}$$

in such a way that $H_{\rm ss}$ is *linear* in $H^p = \begin{pmatrix} \tilde{H} & 0 \\ 0 & H \end{pmatrix}$:

$$H_{\rm ss} = (H^p - \epsilon). \tag{17}$$

Notice that the Hamiltonians H, $\tilde{H}$ in the diagonal of H^p are intertwined by $B^\dagger$:

$$\begin{aligned} &\tilde{H}B^\dagger = B^\dagger H, \\ &H = -\frac{1}{2}\frac{d^2}{dx^2} + V(x), \\ &\tilde{H} = -\frac{1}{2}\frac{d^2}{dx^2} + \tilde{V}(x). \end{aligned} \tag{18}$$

This is equivalent to say that $V,\ \tilde{V}$ and α_1 are related through:

$$\begin{aligned} &\alpha_1'(x,\epsilon) + \alpha_1^2(x,\epsilon) = 2[V(x) - \epsilon], \\ &\tilde{V}(x) = V(x) - \alpha_1'(x,\epsilon). \end{aligned} \tag{19}$$

The real number ϵ is called *factorization energy* (although this does not mean that it should be a physical energy of H):

$$\begin{aligned} &H = BB^\dagger + \epsilon, \\ &\tilde{H} = B^\dagger B + \epsilon. \end{aligned} \tag{20}$$

Let us denote by $\psi_n(x)$ the eigenfunctions of H, $H\psi_n(x) = E_n\psi_n(x)$. From the intertwining relationship (18) it is simple to get the eigenfunctions of $\tilde{H}$:

$$\tilde{\psi}_n(x) = \frac{B^\dagger \psi_n(x)}{\sqrt{E_n - \epsilon}}, \quad \tilde{H}\tilde{\psi}_n(x) = E_n \tilde{\psi}_n(x). \tag{21}$$

There is an eigenfunction $\tilde{\psi}_\epsilon(x)$ additional to those of (21), which satisfies:

$$B\tilde{\psi}_\epsilon(x) = 0 \quad \Rightarrow \quad \tilde{\psi}_\epsilon(x) \propto \exp\left[-\int_0^x \alpha_1(y,\epsilon)dy\right]. \tag{22}$$

The corresponding eigenvalue is ϵ:

$$\tilde{H}\tilde{\psi}_\epsilon(x) = \epsilon\tilde{\psi}_\epsilon(x). \tag{23}$$

We conclude that, departing from H, a new solvable Hamiltonian $\tilde{H}$ with eigenfunctions $\{\tilde{\psi}_\epsilon(x), \tilde{\psi}_n(x)\}$ and eigenvalues $\{\epsilon, E_n\}$ can be generated. Let us remark the importance of the value of ϵ. It turns out that a necessary condition in order to avoid supplementary singularities of $\tilde{V}(x)$ with respect to those of $V(x)$ is that ϵ is less than or equal to the ground state energy E_0 of H.

4. Higher order supersymmetric quantum mechanics. Let us construct now a sequence of Hamiltonians $H_1, \dots, H_m$ departing from H_0 as follows [11]:

$$\begin{aligned} &H_i A_i^\dagger = A_i^\dagger H_{i-1}, \\ &H_i = -\frac{1}{2}\frac{d^2}{dx^2} + V_i(x), \\ &A_i^\dagger = \frac{1}{\sqrt{2}}\left[-\frac{d}{dx} + \alpha_i(x,\epsilon_i)\right]. \end{aligned} \tag{24}$$

Taking into account equations (18-19), we get once again a Riccati equation and the corresponding expression of the potential at the i-th step in terms of the $(i-1)$-th potential:

$$\begin{aligned} &\alpha_i'(x,\epsilon_i) + \alpha_i^2(x,\epsilon_i) = 2\left[V_{i-1}(x) - \epsilon_i\right], \\ &V_i(x) = V_{i-1}(x) - \alpha_i'(x,\epsilon_i). \end{aligned} \tag{25}$$

The factorized expressions arising form the iterative procedure become:

$$\begin{aligned} &H_0 = A_1 A_1^\dagger + \epsilon_1 \\ &H_i = A_i^\dagger A_i + \epsilon_i = A_{i+1}A_{i+1}^\dagger + \epsilon_{i+1} \\ &H_m = A_m^\dagger A_m + \epsilon_m \end{aligned} \tag{26}$$

The corresponding eigenfunctions of H_i are:

$$\begin{cases} \psi_{\epsilon_i}^{(i)}(x) \propto \exp[-\int_0^x \alpha_i(y,\epsilon_i)dy], \\ \psi_{\epsilon_{i-1}}^{(i)}(x) = \frac{A_i^\dagger \psi_{\epsilon_{i-1}}^{(i-1)}(x)}{\sqrt{\epsilon_{i-1}-\epsilon_i}}, \\ \qquad\qquad \vdots \\ \psi_{\epsilon_1}^{(i)}(x) = \frac{A_i^\dagger \dots A_2^\dagger \psi_{\epsilon_1}^{(1)}(x)}{\sqrt{(\epsilon_1-\epsilon_2)\dots(\epsilon_1-\epsilon_i)}}, \\ \psi_n^{(i)}(x) = \frac{A_i^\dagger \dots A_1^\dagger \psi_n^{(0)}(x)}{\sqrt{(E_n-\epsilon_1)\dots(E_n-\epsilon_i)}}, \end{cases} \tag{27}$$

The eigenvalues become $\{\epsilon_k, E_n, k = 1,\dots,i, n = 0,\dots\}$.

Let us make now in (14-15) the following identifications:

$$\begin{aligned} B &= A_1 \dots A_m, \\ B^\dagger &= A_m^\dagger \dots A_1^\dagger, \\ H &= H_0, \\ \tilde{H} &= H_m. \end{aligned} \tag{28}$$

With such a choice one obtains the higher order supersymmetric quantum mechanics in which $H_{\rm ss}$ is a m-th order polynomial in H^p [11,18-25]:

$$H_{\rm ss} = (H^p - \epsilon_1)\dots(H^p - \epsilon_m). \tag{29}$$

By simplicity, we have ordered the factorization energies as $\epsilon_m < \epsilon_{m-1} < \cdots < \epsilon_1 \leq E_0$. The final potential $\tilde{V}(x)$ in terms of the initial one is given by:

$$\tilde{V}(x) = V(x) - \sum_{i=1}^{m} \alpha_i'(x,\epsilon_i). \tag{30}$$

Thus, the point consists in looking for the solutions to the Riccati equation (25) for the various values of i. It turns out that the i-th superpotential $\alpha_i(x,\epsilon_i)$ can be algebraically determined from the $(i-1)$-th at ϵ_i and ϵ_{i-1} [22-25]:

$$\alpha_i(x,\epsilon_i) = -\alpha_{i-1}(x,\epsilon_{i-1}) - \frac{2(\epsilon_{i-1}-\epsilon_i)}{\alpha_{i-1}(x,\epsilon_{i-1}) - \alpha_{i-1}(x,\epsilon_i)}. \tag{31}$$

We arrive, finally, to the following conclusion: the m solutions $\alpha_1(x,\epsilon_i), i = 1,\dots,m$ to the initial Riccati equation (19) determine completely, by means of just an algebraic composition procedure involving equation (31), the final potential (30).

4.1. The harmonic oscillator. By taking $V(x) = x^2/2$, it is simple to find the general solution to (19) for an arbitrary fixed value of ϵ [14,17]:

$$\alpha_1(x,\epsilon) = -x + \frac{d}{dx}\Bigg\{ \ln\left[{}_1F_1\left(\tfrac{1-2\epsilon}{4}, \tfrac{1}{2}; x^2\right) + 2\nu \frac{\Gamma(\frac{3-2\epsilon}{4})}{\Gamma(\frac{1-2\epsilon}{4})}\, x\, {}_1F_1\left(\tfrac{3-2\epsilon}{4}, \tfrac{3}{2}; x^2\right)\right]\Bigg\}. \tag{32}$$

In the first order supersymmetric quantum mechanics, we must have $|\nu| < 1$ in order to avoid singularities in $\tilde{V}(x)$. The corresponding restriction for ν in the higher order case is, in general, different from the one just pointed out [22]. The eigenfunctions of the new potentials (30) are given by (27). The corresponding spectrum takes the form $\{\epsilon_i, E_n = n + 1/2, i = 1,\dots,m, n = 0,1,\dots\}$, i.e., there is one infinite ladder of equally spaced levels starting from 1/2 plus m finite ones of length one and placed at arbitrary positions $\epsilon_i,\ i = 1,\dots,m$ below 1/2.

5. Polynomial Heisenberg algebras and HSUSY. An obvious constructions for a *natural* pair of annihilation and creation operators associated to the HSUSY partners $\tilde{H}$ of the oscillator can be seen from figure 3 [11,27-28]:

$$D = B^\dagger a B, \qquad D^\dagger = B^\dagger a^\dagger B, \tag{33}$$

where a and $a^\dagger$ are the annihilation and creation operators of the harmonic oscillator. It is simple to show that $D,\ D^\dagger$ and $\tilde{H}$ close a polynomial algebra of $(2m)$-th order of kind (7-9) [11]:

$$\begin{aligned} &[\tilde{H}, D] = -D, \\ &[\tilde{H}, D^\dagger] = D^\dagger, \\ &[D, D^\dagger] = N(\tilde{H}+1) - N(\tilde{H}). \end{aligned} \tag{34}$$

The generalized number operator $N(\tilde{H})$ reads:

$$N(\tilde{H}) \equiv D^\dagger D = \left(\tilde{H} - \frac{1}{2}\right) \prod_{i=1}^{m} \left(\tilde{H} - \epsilon_i - 1\right) \left(\tilde{H} - \epsilon_i\right).$$

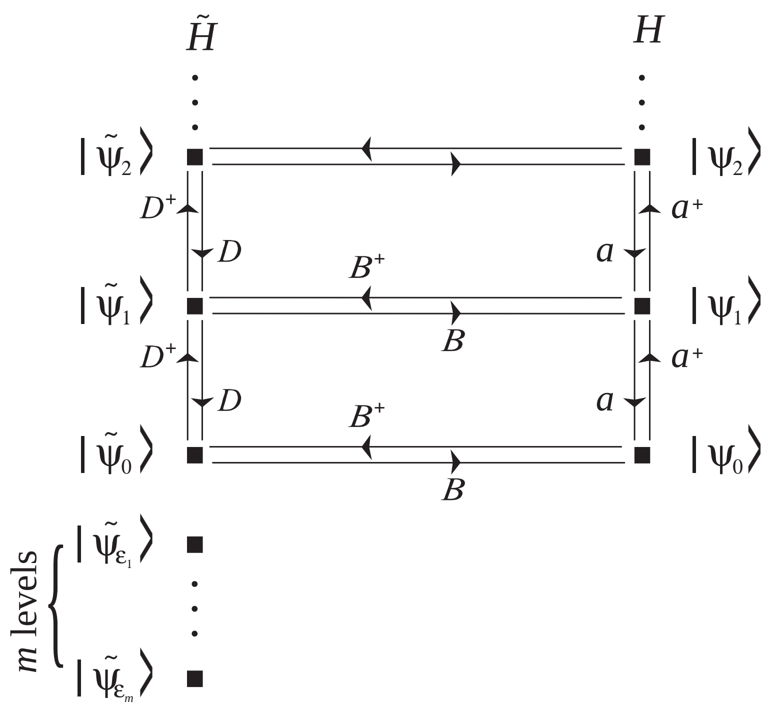

Figure 3: The spectra of the HSUSY partners of the harmonic oscillator. There is one infinite ladder starting from $E_0 = 1/2$ and m finite ones starting and ending at $\epsilon_i, i = 1, \dots, m$.

We notice that the eigenstates $|\tilde{\psi}_{\epsilon_i}\rangle$ of $\tilde{H}$ are annihilated by both D and $D^\dagger$. Moreover, any eigenstate $|\tilde{\psi}_n\rangle,\ n = 1, 2, \dots$ of $\tilde{H}$ is connected with its two neighbours $|\tilde{\psi}_{n-1}\rangle$ and $|\tilde{\psi}_{n+1}\rangle$ by means of D and $D^\dagger$ and the extremal state $|\tilde{\psi}_0\rangle$ is just connected with $|\tilde{\psi}_1\rangle$ by means of $D^\dagger$ because it is annihilated by D.

We have provided thus a non-trivial realization of the Polynomial Heisenberg algebra (7-9). Next, we will look for a variant of the annihilation and creation operators whose action onto the basis vectors $\{|\tilde{\psi}_{\epsilon_i}\rangle, |\tilde{\psi}_n\rangle\}$ becomes similar as for the harmonic oscillator.

6. Linearized polynomial Heisenberg algebra. The pair of annihilation and creation operators D_L and $D_L^\dagger$ acting onto $|\tilde{\psi}_n\rangle,\ n=0,1,\dots$ as the Heisenberg-Weyl generators take the form [11,29]:

$$
\begin{aligned}
D_L &= B^\dagger \left[\prod_{i=1}^{m}(N-\epsilon_i+\frac{1}{2})(N-\epsilon_i+\frac{3}{2})\right]^{-\frac{1}{2}} aB, \\
D_L^\dagger &= B^\dagger a^\dagger \left[\prod_{i=1}^{m}(N-\epsilon_i+\frac{1}{2})(N-\epsilon_i+\frac{3}{2})\right]^{-\frac{1}{2}} B,
\end{aligned}
\tag{35}
$$

$$
\begin{aligned}
D_L|\tilde{\psi}_n\rangle &= \sqrt{n}|\tilde{\psi}_{n-1}\rangle, \\
D_L^\dagger|\tilde{\psi}_n\rangle &= \sqrt{n+1}|\tilde{\psi}_{n+1}\rangle, \\
[D_L, D_L^\dagger]|\tilde{\psi}_n\rangle &= |\tilde{\psi}_n\rangle.
\end{aligned}
\tag{36}
$$

Let us notice that the expressions (35) for D_L and $D_L^\dagger$ are more involved than the corresponding ones for D and $D^\dagger$ in (33). However, D_L and $D_L^\dagger$ act onto $\{|\tilde{\psi}_n\rangle, n=0,1,\dots\}$ simpler than D and $D^\dagger$ do. This property arises again for the corresponding coherent states, a point which has been discussed in detail elsewhere [11].

7. Conclusions and outlook. The HSUSY QM is a powerful tool in order to generate solvable potentials from a given initial one. The point reduces to find solutions to the initial Riccati equation (19) for a set of factorization energies $\{\epsilon_i,\ i=1,\dots,m\}$. In the oscillator case it is possible to iterate the process an arbitrary number of times because it is known the general solution to (19) in the full range $\epsilon < E_0$. The HSUSY partners of the oscillator have a pair of natural annihilation and creation operators generating a $(2m)$-th order polynomial Heisenberg algebra. This kind of algebras can be linearized on the subspace spanned by the eigenvectors associated to $E_n = n+\frac{1}{2},\ n=0,1,\dots$ It has been shown as well that D_L and $D_L^\dagger$ represent better than D and $D^\dagger$ the role played by a and $a^\dagger$ for the oscillator potential.

As a final remark, let us notice that the polynomial Heisenberg algebras can be realized as well by the harmonic oscillator Hamiltonian and some deformed versions of the standard annihilation and creation operators [11]. This procedure can be considered artificial in the sense that a complicated algebraic structure is constructed for a system which has from the very beginning an inherently simpler algebra. We conclude that the HSUSY partners of the oscillator provide the simplest systems for which the polynomial Heisenberg algebras arise in a natural way. Let us notice, however, that systems more general than the HSUSY partners of the oscillator exist which have this kind of deformed algebraic structures [6,12,26,30]. The study of such general systems is a subject of current research which will continue in the near future.

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